Stability of networks of infinite-dimensional systems

Andrii Mironchenko

Faculty of Mathematics and Computer Science
University of Passau

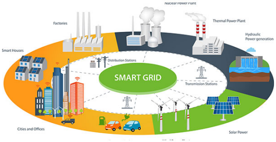
IFAC World Congress 2020

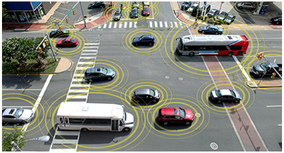
Pre-Conference Workshop
Input-to-state stability and control of infinite-dimensional systems

11 July 2020

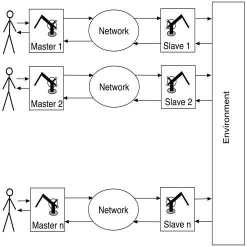
www.mironchenko.com

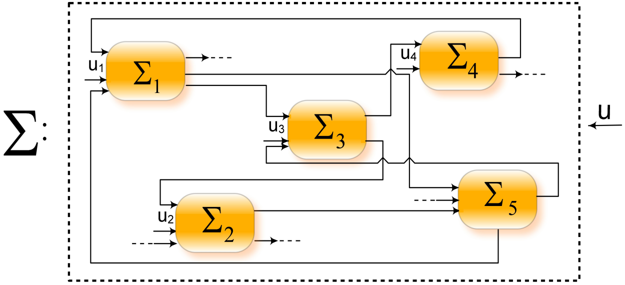
Motivation: Large-scale systems



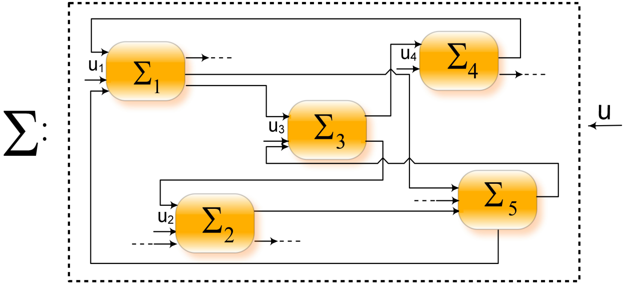


- Emerging technologies such as 5G, IoT, Clouds, make the networks larger and larger.
- Components may belong to different system classes
- The size is either fixed or unknown
- Safety and reliability need to be analytically verified





Under which conditions
an interconnection of stable systems is stable?



Under which conditions
an interconnection of stable systems is stable?

Outline

- 1 part of the talk: couplings of 2 systems
- 2 part of the talk: infinite networks

Class of systems

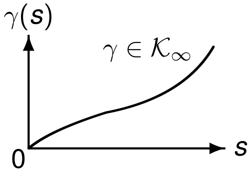
$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X.$$

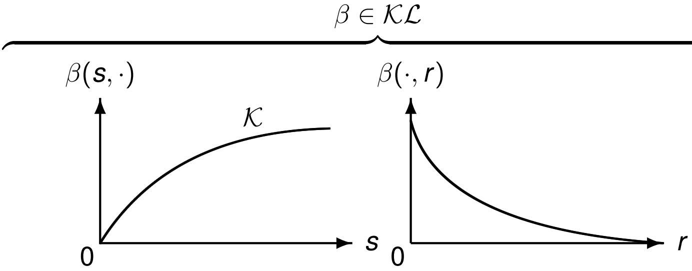
- X = State space
- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.

$x \in C([0, T], X)$ is a mild solution iff

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

Comparison functions





Definition (Sontag, 1989, for ODEs)

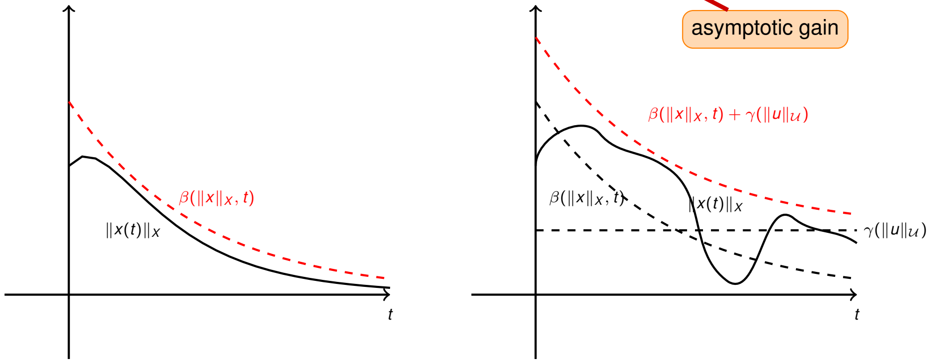
$$\text{ISS} \quad :\Leftrightarrow \quad \|x(t)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

increasing in $\|x\|_X$

decreasing to 0 in t

$\gamma(0) = 0$, increasing

asymptotic gain



(a) $u \equiv 0$ (b) $u \not\equiv 0$

Integral input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGAS))

0-UGAS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}$: $\forall x \in X, \forall t \geq 0$

$$\|\phi(t,x,0)\|_X \leq \beta(\|x\|_X, t).$$

Definition (Integral input-to-state stability (iISS))

iISS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}, \theta, \mu \in \mathcal{K}$: $\forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \theta\Big(\int_0^t \mu(\|u(s)\|_U) ds\Big).$$

Integral input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGAS))

0-UGAS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}$: $\forall x \in X, \forall t \geq 0$

$$\|\phi(t,x,0)\|_X \leq \beta(\|x\|_X, t).$$

Definition (Integral input-to-state stability (iISS))

iISS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}, \theta, \mu \in \mathcal{K}$: $\forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \theta\Big(\int_0^t \mu(\|u(s)\|_U) ds\Big).$$

Overview of the infinite-dimensional ISS theory

- Karafyllis, Krstic. Input-to-state stability for PDEs. Springer, 2019.
- M., Prieur. Input-to-state stability of infinite-dimensional systems: recent results and open questions. *To appear in SIAM Review*, 2020.

Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)).$$

Definition

$V : X \to \mathbb{R}_+$ is an iISS-Lyapunov function iff $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$ and $\sigma, \alpha \in \mathcal{K}$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $\dot{V}_u(x) \leq -\alpha(V(x)) + \sigma(\|u(0)\|_U)$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}\big(V(\phi(t, x, u)) - V(x)\big).$$

$\alpha \in \mathcal{K}_\infty \Rightarrow V$ is an ISS-Lyapunov function.

Theorem

$\exists$ *ISS/iISS Lyapunov function* $\Rightarrow$ *ISS/iISS.*

Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)).$$

Definition

$V : X \to \mathbb{R}_+$ is an iISS-Lyapunov function iff $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$ and $\sigma, \alpha \in \mathcal{K}$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $\dot{V}_u(x) \leq -\alpha(V(x)) + \sigma(\|u(0)\|_U)$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x)).$$

$\alpha \in \mathcal{K}_\infty \Rightarrow V$ is an ISS-Lyapunov function.

Theorem

$\exists$ *ISS/iISS Lyapunov function* $\Rightarrow$ *ISS/iISS.*

What about coupled systems?

Interconnections of two iISS systems

$$\Sigma : \begin{cases} \Sigma_1 : & \dot{x}_1 = A_1 x_1 + f_1(x_1, x_2, u), \;\; x_1 \in X_1 \\ \Sigma_2 : & \dot{x}_2 = A_2 x_2 + f_2(x_1, x_2, u), \;\; x_2 \in X_2 \end{cases}$$

iISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is iISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $\dot{V}_1(x_1) \leq -\alpha_1(\|x_1\|_{X_1}) + \sigma_1(\|x_2\|_{X_2}) + \kappa_1(\|u(0)\|_U)$,
- $\dot{V}_2(x_2) \leq -\alpha_2(\|x_2\|_{X_2}) + \sigma_2(\|x_1\|_{X_1}) + \kappa_2(\|u(0)\|_U)$,

Interconnections of two iISS systems

$$\Sigma : \begin{cases} \Sigma_1 : & \dot{x}_1 = A_1 x_1 + f_1(x_1, x_2, u), \;\; x_1 \in X_1 \\ \Sigma_2 : & \dot{x}_2 = A_2 x_2 + f_2(x_1, x_2, u), \;\; x_2 \in X_2 \end{cases}$$

iISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is iISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $\dot{V}_1(x_1) \leq -\alpha_1(\|x_1\|_{X_1}) + \sigma_1(\|x_2\|_{X_2}) + \kappa_1(\|u(0)\|_U)$,
- $\dot{V}_2(x_2) \leq -\alpha_2(\|x_2\|_{X_2}) + \sigma_2(\|x_1\|_{X_1}) + \kappa_2(\|u(0)\|_U)$,

Lyapunov gains

- $gain_{\Sigma_2 \to \Sigma_1} := \alpha_1^{\ominus} \circ \sigma_1$
- $gain_{\Sigma_1 \to \Sigma_2} := \alpha_2^{\ominus} \circ \sigma_2$

$$\omega^{\ominus}(s) := \begin{cases} \omega^{-1}(s) & \text{, if } s \in \operatorname{Im} \omega \\ +\infty & \text{, otherwise} \end{cases}$$

Small-gain theorem for 2 interconnected iISS systems

Theorem (A. Mironchenko, H. Ito, SICON, 2015)

Let:

- $V_i(x_i) = \psi_i(\|x_i\|_{X_i})$
- $\exists c > 1$: $\forall s \in \mathbb{R}_+$:

$$\underbrace{\alpha_1^{\ominus} \circ c\sigma_1}_{\approx gain_{\Sigma_2 \to \Sigma_1}} \circ \underbrace{\alpha_2^{\ominus} \circ c\sigma_2}_{\approx gain_{\Sigma_1 \to \Sigma_2}}(s) \leq s.$$

$\Rightarrow$ *Σ is iISS.*

If additionally

- $\alpha_i \in \mathcal{K}_\infty$ *for* $i = 1, 2$ $\quad\Rightarrow\quad$ *Σ is ISS.*

$$\textit{iISS-LF:} \qquad V(x) = \int_0^{V_1(x_1)} \lambda_1(s)ds + \int_0^{V_2(x_2)} \lambda_2(s)ds.$$

- M., Ito. *Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach.* SICON, 2015.

Example

$$\begin{cases} \frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\left(\frac{\partial x_2}{\partial l}\right)^2 + \left(\frac{x_1^2}{1+x_1^2}\right)^{\frac{1}{2}}, \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases}$$

Example

$$
\begin{cases}
\frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\
x_1(0,t) = x_1(\pi,t) = 0; \\
\frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\left(\frac{\partial x_2}{\partial l}\right)^2 + \left(\frac{x_1^2}{1+x_1^2}\right)^{\frac{1}{2}}, \\
x_2(0,t) = x_2(\pi,t) = 0.
\end{cases}
$$

For what a, b is this system UGAS?

Example

$$
\begin{cases}
\frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\
x_1(0,t) = x_1(\pi,t) = 0; \\
\frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\left(\frac{\partial x_2}{\partial l}\right)^2 + \left(\frac{x_1^2}{1+x_1^2}\right)^{\frac{1}{2}}, \\
x_2(0,t) = x_2(\pi,t) = 0.
\end{cases}
$$

For what a, b is this system UGAS?

$$X_1 := L_2(0,\pi) \qquad X_2 := H_0^1(0,\pi)$$

Strategy

1. x_1-subsystem is iISS
2. x_2-subsystem is ISS
3. Interconnection is UGAS

x_1-subsystem is iISS

- iISS-LF for Σ_1: $V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right)$
- Lie derivative of V_1: $\dot{V}_1(x_1) \leq \underbrace{-\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}}_{\alpha_1(\|x_1\|_{L_2(0,\pi)})} + \underbrace{8\|x_2\|^4_{H^1_0(0,\pi)}}_{\sigma_1(\|x_2\|_{H^1_0(0,\pi)})}$

x_1-subsystem is iISS

- iISS-LF for Σ_1: $V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right)$
- Lie derivative of V_1: $\dot V_1(x_1) \leq \underbrace{-\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}}_{\alpha_1(\|x_1\|_{L_2(0,\pi)})} + \underbrace{8\|x_2\|^4_{H^1_0(0,\pi)}}_{\sigma_1(\|x_2\|_{H^1_0(0,\pi)})}$

x_2-subsystem is ISS

- ISS-LF for Σ_2: $V_2(x_2) = \int_0^\pi \left(\frac{\partial x_2}{\partial l}\right)^2 dl = \|x_2\|^2_{H^1_0(0,\pi)}$
- Lie derivative of V_2:

$$\dot V_2 \leq \underbrace{-2\left(1 - a - \tfrac{\omega}{2}\right)\|x_2\|^2_{H^1_0(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H^1_0(0,\pi)}}_{\alpha_2(\|x_2\|_{H^1_0(0,\pi)})} + \underbrace{\frac{\pi}{\omega}\left(\frac{\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}\right)}_{\sigma_2(\|x_1\|_{L_2(0,\pi)})}$$

x_1-subsystem is iISS

- iISS-LF for Σ_1: $V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right)$
- Lie derivative of V_1: $\dot V_1(x_1) \leq \underbrace{- \frac{2\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}}_{\alpha_1(\|x_1\|_{L_2(0,\pi)})} + \underbrace{8\|x_2\|^4_{H^1_0(0,\pi)}}_{\sigma_1(\|x_2\|_{H^1_0(0,\pi)})}$

x_2-subsystem is ISS

- ISS-LF for Σ_2: $V_2(x_2) = \int_0^\pi \left(\frac{\partial x_2}{\partial l}\right)^2 dl = \|x_2\|^2_{H^1_0(0,\pi)}$
- Lie derivative of V_2:

$$\dot V_2 \leq \underbrace{-2\left(1 - a - \tfrac{\omega}{2}\right)\|x_2\|^2_{H^1_0(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H^1_0(0,\pi)}}_{\alpha_2(\|x_2\|_{H^1_0(0,\pi)})} + \underbrace{\frac{\pi}{\omega}\left(\frac{\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}}\right)}_{\sigma_2(\|x_1\|_{L_2(0,\pi)})}$$

Condition for UGAS: for some $c > 0$, for all $s \in \mathbb{R}_+$

$$\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s$$

x_1-subsystem is iISS

- iISS-LF for Σ_1: $V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right)$
- Lie derivative of V_1: $\dot V_1(x_1) \leq \underbrace{-\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}}_{\alpha_1(\|x_1\|_{L_2(0,\pi)})} + \underbrace{8\|x_2\|^4_{H^1_0(0,\pi)}}_{\sigma_1(\|x_2\|_{H^1_0(0,\pi)})}$

x_2-subsystem is ISS

- ISS-LF for Σ_2: $V_2(x_2) = \int_0^\pi \left(\frac{\partial x_2}{\partial l}\right)^2 dl = \|x_2\|^2_{H^1_0(0,\pi)}$
- Lie derivative of V_2:

$$\dot V_2 \leq \underbrace{-2\left(1 - a - \tfrac{\omega}{2}\right)\|x_2\|^2_{H^1_0(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H^1_0(0,\pi)}}_{\alpha_2(\|x_2\|_{H^1_0(0,\pi)})} + \underbrace{\frac{\pi}{\omega}\left(\frac{\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}}\right)}_{\sigma_2(\|x_1\|_{L_2(0,\pi)})}$$

Condition for UGAS

$$a + \frac{3\pi^2}{b} < 1, \quad b > 0.$$

Interim Conclusion

Lyapunov-based small-gain approach for stability analysis of 2 coupled systems

- Find (i)ISS Lyapunov functions for subsystems
- Compute the gains
- Check the small-gain condition

Outlook

- The results have been shown for systems with in-domain coupling
- But they can be extended also to the case of boundary couplings
- The complexity on this way is
  - to establish well-posedness of the coupled system
  - to compute (i)ISS Lyapunov functions for subsystems.
    Here non-coercive ISS Lyapunov functions can be useful.

- M., Ito. *Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach.* SICON, 2015.

Nonlinear ISS small-gain theorems: Literature overview

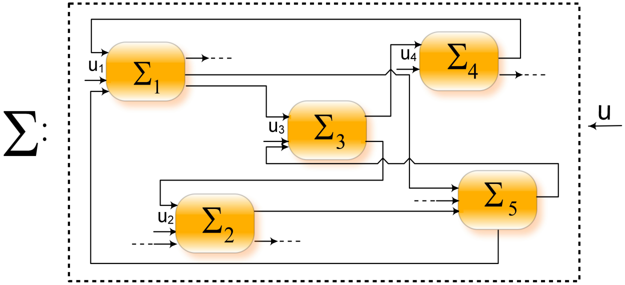
- Small-gain theorems for 2 (i)ISS ODE systems
  [Jiang, Teel, Praly, 1994], [Jiang, Mareels, Wang, 1996], [Ito, 2006] . . .
- Small-gain theorems for n ISS ODE systems
  [Dashkovskiy, Rüffer, Wirth, 2007, 2010], [Ito, Jiang, 2009], [Dashkovskiy, Ito, Wirth, 2011] . . .
- Extensions to n ISS time-delay systems
  [Polushin, Tayebi, Marquez, 2006], [Polushin, Dashkovskiy, Takhmar, Patel, 2013], [Tiwari, Wang, Jiang, 2009, 2012], [Dashkovskiy, Kosmykov, Mironchenko, Naujok, 2012], . . .
- Extensions to n ISS infinite-dimensional systems
  [Dashkovskiy, Mironchenko, 2013], [Mironchenko, Ito, 2015], [Bao, Liu, Jiang, Zhang, 2018], . . .

Nonlinear ISS small-gain theorems: Literature overview

- Small-gain theorems for 2 (i)ISS ODE systems
  [Jiang, Teel, Praly, 1994], [Jiang, Mareels, Wang, 1996], [Ito, 2006] . . .
- Small-gain theorems for n ISS ODE systems
  [Dashkovskiy, Rüffer, Wirth, 2007, 2010], [Ito, Jiang, 2009], [Dashkovskiy, Ito, Wirth, 2011] . . .
- Extensions to n ISS time-delay systems
  [Polushin, Tayebi, Marquez, 2006], [Polushin, Dashkovskiy, Takhmar, Patel, 2013], [Tiwari, Wang, Jiang, 2009, 2012], [Dashkovskiy, Kosmykov, Mironchenko, Naujok, 2012], . . .
- Extensions to n ISS infinite-dimensional systems
  [Dashkovskiy, Mironchenko, 2013], [Mironchenko, Ito, 2015], [Bao, Liu, Jiang, Zhang, 2018], . . .
- (Parallel development) Spatially invariant networks
  [Bamieh, Paganini, Dahleh, 2002], [Bamieh, Voulgaris, 2005], [Besselink, Johansson, 2017], [Curtain, Iftime, Zwart, 2009], . . .

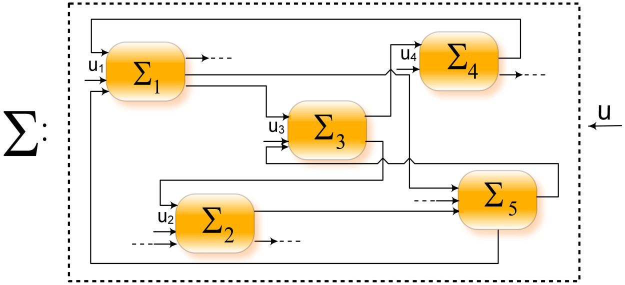
Nonlinear ISS small-gain theorems: Literature overview

- Small-gain theorems for 2 (i)ISS ODE systems
  [Jiang, Teel, Praly, 1994], [Jiang, Mareels, Wang, 1996], [Ito, 2006] ...
- Small-gain theorems for n ISS ODE systems
  [Dashkovskiy, Rüffer, Wirth, 2007, 2010], [Ito, Jiang, 2009], [Dashkovskiy, Ito, Wirth, 2011] ...
- Extensions to n ISS time-delay systems
  [Polushin, Tayebi, Marquez, 2006], [Polushin, Dashkovskiy, Takhmar, Patel, 2013], [Tiwari, Wang, Jiang, 2009, 2012], [Dashkovskiy, Kosmykov, Mironchenko, Naujok, 2012], ...
- Extensions to n ISS infinite-dimensional systems
  [Dashkovskiy, Mironchenko, 2013], [Mironchenko, Ito, 2015], [Bao, Liu, Jiang, Zhang, 2018], ...
- (Parallel development) Spatially invariant networks
  [Bamieh, Paganini, Dahleh, 2002], [Bamieh, Voulgaris, 2005], [Besselink, Johansson, 2017], [Curtain, Iftime, Zwart, 2009], ...
- (Partial extensions) Small-gain theory for infinite networks
  [Dashkovskiy, Pavlichkov, 2020], [Dashkovskiy, Mironchenko, Schmid, Wirth, 2019], [Kawan, Mironchenko, Swikir, Noroozi, Zamani, 2019], ...



Today:

General nonlinear ISS small-gain theorem for infinite networks.



We develop such conditions for:

- Finite and infinite networks
- Subsystems of any dimension
- Subsystems of any type (ODEs, PDEs, delay and switched systems, etc.)
- Couplings of any type (in-domain or boundary couplings)
- No assumption of spatial invariance

Class of systems

Definition

The triple $\Sigma = (X, \mathcal{U}, \phi)$, $\quad \phi : \mathbb{R}_+ \times X \times \mathcal{U} \to X$ is called control system, if:

(Σ1) Forward-completeness: for every $x \in X$, $u \in \mathcal{U}$ and for all $t \geq 0$ the value $\phi(t, x, u) \in X$ is well-defined.

(Σ2) Continuity: for each $(x, u) \in X \times \mathcal{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous.

(Σ3) Cocycle property: for all $t, h \geq 0$, for all $x \in X$, $u \in \mathcal{U}$ we have

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

Class of systems

Definition

The triple $\Sigma = (X, \mathcal{U}, \phi)$, $\phi : \mathbb{R}_+ \times X \times \mathcal{U} \to X$ is called control system, if:

(Σ1) Forward-completeness: for every $x \in X$, $u \in \mathcal{U}$ and for all $t \geq 0$ the value $\phi(t, x, u) \in X$ is well-defined.

(Σ2) Continuity: for each $(x, u) \in X \times \mathcal{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous.

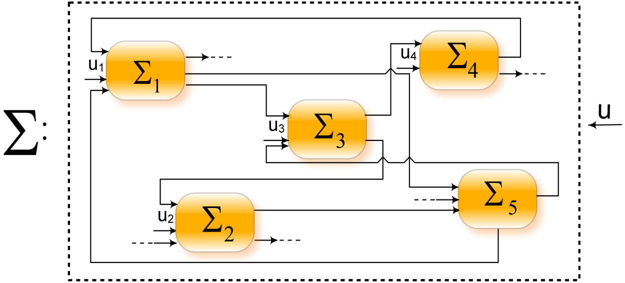
(Σ3) Cocycle property: for all $t, h \geq 0$, for all $x \in X$, $u \in \mathcal{U}$ we have

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

Examples

- Ordinary differential equations
- Evolution Partial differential equations with Lipschitz nonlinearities
- Broad classes of boundary control systems
- Time-delay systems
- Heterogeneous systems with distinct components

Interconnections of abstract systems



- We want to interconnect heterogeneous systems (PDEs, delays, ODEs)
- We want to allow both boundary and in-domain couplings.
- We assume that the couplings are well-defined.

To model such general couplings we use (and extend from 2 to ∞ systems)

- Karafyllis, Jiang. *A small-gain theorem for a wide class of feedback systems with control applications*, SICON, 2007.

Input-to-state stability

Definition (Sontag, 1989, for ODEs)

$$\text{ISS} \quad :\Leftrightarrow \quad \|x(t)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

Definition (Uniform global stability)

$$\text{UGS} \quad :\Leftrightarrow \quad \|x(t)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

Input-to-state stability

Definition (Sontag, 1989, for ODEs)

$$\text{ISS} \quad :\Leftrightarrow \quad \|x(t)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

Definition (Uniform global stability)

$$\text{UGS} \quad :\Leftrightarrow \quad \|x(t)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

Definition (Bounded input uniform asymptotic gain property)

bUAG $\quad :\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty\colon \forall r, \varepsilon > 0 \ \exists \tau = \tau(\varepsilon, r) > 0$ such that

$$\|x\|_X \le r \ \wedge \|u\|_{\mathcal{U}} \le r \wedge t \ge \tau \quad \Rightarrow \quad \|x(t)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

Input-to-state stability

Definition (Sontag, 1989, for ODEs)

$$\text{ISS} \quad :\Leftrightarrow \quad \|x(t)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

Definition (Uniform global stability)

$$\text{UGS} \quad :\Leftrightarrow \quad \|x(t)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

Definition (Bounded input uniform asymptotic gain property)

bUAG $:\Leftrightarrow$ $\exists \gamma \in \mathcal{K}_\infty$: $\forall r, \varepsilon > 0$ $\exists \tau = \tau(\varepsilon, r) > 0$ such that

$$\|x\|_X \le r \ \wedge \|u\|_{\mathcal{U}} \le r \wedge t \ge \tau \quad \Rightarrow \quad \|x(t)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

Lemma (follows from a much stronger result in M., Wirth, IEEE TAC, 2018)

Let Σ be a forward complete control system.

$$\textit{ISS} \quad \Leftrightarrow \quad \textit{UGS} \wedge \textit{bUAG}$$

Infinite networks: ISS of subsystems

Definition (ISS for a subsystem Σ_i)

Σ_i is ISS (in semimaximum formulation) if: $\exists\ \gamma_{ij}, \gamma_i \in \mathcal{K}_\infty$, $\exists \beta_i \in \mathcal{KL}$ s.t.:
$\forall x_i,\ u,\ t,\ w_{\neq i} := (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots)$ we have:

$$\|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} \leq \beta_i\left(\|x_i\|_{X_i}, t\right) + \sup_{j \neq i} \gamma_{ij}\left(\|w_j\|_{[0,t]}\right) + \gamma_i\left(\|u\|_{\mathcal{U}}\right).$$

- State space: $X := X_1 \times X_2 \times \ldots$: $\quad \|x\|_X := \sup_{j \in \mathbb{N}} \left\{\|x_j\|_{X_j}\right\} < \infty$.
- Internal inputs to the i-th subsystems: $\quad \|x\|_{X_{\neq i}} := \sup_{j \in \mathbb{N},\ j \neq i} \left\{\|x_j\|_{X_j}\right\}$.
- Infinite gain matrix: $\Gamma := (\gamma_{ij})_{i,j \in \mathbb{N}}$
- Gain operator: $\Gamma_\otimes : \ell_\infty^+ \to \ell_\infty^+$

$$\Gamma_\otimes(s) := \left(\sup_{j=1}^{\infty} \gamma_{1j}(s_j),\ \sup_{j=1}^{\infty} \gamma_{nj}(s_j), \ldots\right)^T, \quad s = (s_1, s_2, \ldots)^T \in \ell_\infty^+.$$

Which properties of $\Gamma_\otimes$ ensure ISS of the network?

Trajectory-based small-gain theorem for infinite networks

Definition

The monotone nonlinear operator $A : X \to X$ has the monotone limit property (MLIM) if $\exists$ $\xi \in \mathcal{K}_\infty$: $\forall \varepsilon > 0$, $\forall u \in \ell_\infty(\mathbb{Z}_+, X^+)$ and any monotone solution $x(\cdot) = (x(k))_{k\in\mathbb{Z}_+}$ of

$$x(k+1) \le A(x(k)) + u(k), \quad k \in \mathbb{Z}_+,$$

satisfying $x(\cdot) \subset X^+$ it holds that

$$\exists N = N(\varepsilon, u, x(\cdot)) \in \mathbb{Z}_+ : \qquad \|x(N)\|_X \le \varepsilon + \xi(\|u\|_\infty).$$

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ be ISS, $\Sigma = (X, \mathcal{U}, \phi)$ be well-defined and:

1. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$: $\quad \beta_i(r,t) \le \beta(r,t), \quad \gamma_i(r) \le \gamma(r), \quad r \in \mathbb{R}_+,\ t \ge 0,\ i \in \mathbb{N}.$
2. $\Gamma_\otimes$ *has monotone limit property*

Then Σ is ISS.

Trajectory-based small-gain theorem for infinite networks

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let

1. $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ *be ISS*
2. $\Sigma = (X, \mathcal{U}, \phi)$ *be well-defined*
3. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$: $\quad \beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad r \in \mathbb{R}_+,\ t \geq 0,\ i \in \mathbb{N}.$
4. $\Gamma_\otimes$ *has monotone limit property*

Then Σ *is ISS.*

Proof

- Show uniform global stability
- Show uniform asymptotic gain property
- ISS $\quad \Leftrightarrow \quad$ UGS $\wedge$ bUAG

Trajectory-based small-gain theorem for infinite networks

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let

1. $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ *be ISS*
2. $\Sigma = (X, \mathcal{U}, \phi)$ *be well-defined*
3. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$: $\quad \beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad r \in \mathbb{R}_+,\ t \geq 0,\ i \in \mathbb{N}.$
4. $\Gamma_\otimes$ *has monotone limit property*

Then Σ *is ISS.*

Finite ODE networks

- This result was inspired by the small-gain theorem for finite ODE networks [Dashkovskiy, Rüffer, Wirth, 2007].
- However, for ODEs (sufficiently regular) one can use the powerful characterizations of ISS in terms of non-uniform asymptotic gain property [Sontag, Wang, 1996].
- Already for finite networks of infinite-dimensional systems these characterizations do not hold, which was a major challenge on the way to our small-gain theorem for infinite networks

Trajectory-based small-gain theorem for infinite networks

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let

1. $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ *be ISS*
2. $\Sigma = (X, \mathcal{U}, \phi)$ *be well-defined*
3. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$: $\quad \beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad r \in \mathbb{R}_+,\ t \geq 0,\ i \in \mathbb{N}.$
4. $\Gamma_\otimes$ *has monotone limit property*

Then Σ *is ISS.*

Small-gain method for stability analysis of the networks

- Verify ISS of all subsystems and compute the internal gains
- Construct the gain operator $\Gamma_\otimes$
- Verify monotone LIM property for $\quad x(k+1) \leq \Gamma_\otimes(x(k)) + u(k), \quad k \in \mathbb{Z}_+.$

Trajectory-based small-gain theorem for infinite networks

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let

1. $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ *be ISS*
2. $\Sigma = (X, \mathcal{U}, \phi)$ *be well-defined*
3. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$: $\quad \beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad r \in \mathbb{R}_+,\ t \geq 0,\ i \in \mathbb{N}.$
4. $\Gamma_\otimes$ *has monotone limit property*

Then Σ is ISS.

Small-gain method for stability analysis of the networks

- Verify ISS of all subsystems and compute the internal gains
- Construct the gain operator $\Gamma_\otimes$
- Verify monotone LIM property for $\quad x(k+1) \leq \Gamma_\otimes(x(k)) + u(k), \quad k \in \mathbb{Z}_+.$

Verification of the MLIM property is a hard task.

Trajectory-based small-gain theorem for infinite networks

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let

1. $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ *be ISS*
2. $\Sigma = (X, \mathcal{U}, \phi)$ *be well-defined*
3. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$: $\quad \beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad r \in \mathbb{R}_+,\ t \geq 0,\ i \in \mathbb{N}.$
4. $\Gamma_\otimes$ *has monotone limit property*

Then Σ *is ISS.*

Small-gain method for stability analysis of the networks

- Verify ISS of all subsystems and compute the internal gains
- Construct the gain operator $\Gamma_\otimes$
- Verify monotone LIM property for $\quad x(k+1) \leq \Gamma_\otimes(x(k)) + u(k), \quad k \in \mathbb{Z}_+.$

Verification of the MLIM property is a hard task.

Simpler criteria for important special cases are desired.

Small-gain conditions: linear ∞-dim case

Theorem (Criteria for ISS of monotone linear systems (Glück, Kawan, M., 2020))

Let (X, X^+) be an ordered Banach space with a generating, normal and closed cone X^+. Let $A \in L(X)$ be positive, $B \in L(U, X)$, U be a normed linear space. TFAE:

1. *A has monotone limit property.*
2. $x(k+1) = Ax(k) + Bu(k)$ *is ISS*
3. $r(A) < 1$
4. *The uniform small gain condition holds:*

$$\exists \eta \in \mathcal{K}_\infty : \qquad \mathrm{dist}(Ax - x, X^+) \geq \eta(\|x\|_X), \quad x \in X^+.$$

The proof is based on the technique of Fréchet filter powers.

Condition $r(A) < 1$ is tight already for finite networks.

Small-gain conditions: nonlinear n-dim case

Proposition (M., Kawan, Glück, 2020)

Assume that $(X, X^+) = (\mathbb{R}^n, \mathbb{R}^n_+)$ *for some* $n \in \mathbb{N}$*, and* $A : X^+ \to X^+$ *be continuous. TFAE:*

1. *A has monotone limit property.*
2. *The uniform small-gain condition holds:*
$$\exists \eta \in \mathcal{K}_\infty : \qquad \operatorname{dist}\big(A(x) - x, \mathbb{R}^n_+\big) \geq \eta(\|x\|_X) \quad \forall x \in \mathbb{R}^n_+.$$
3. $\exists \eta \in \mathcal{K}_\infty$*:* $\quad A(x) \not\geq x - \eta(\|x\|_X)\mathbf{1} \quad \forall x \in \mathbb{R}^n_+ \backslash \{0\}.$

If $A = \Gamma_\otimes$*, then above conditions are equivalent to*

4. *A satisfies the strong small-gain condition:*
$$\exists \rho \in \mathcal{K}_\infty : \qquad (\mathrm{id} + \rho) \circ A(x) \not\geq x, \quad x \in \mathbb{R}^n_+ \backslash \{0\}.$$

Small-gain conditions: nonlinear n-dim case

Proposition (M., Kawan, Glück, 2020)

Assume that $(X, X^+) = (\mathbb{R}^n, \mathbb{R}^n_+)$ *for some* $n \in \mathbb{N}$, *and* $A : X^+ \to X^+$ *be continuous. TFAE:*

1. *A has monotone limit property.*
2. *The uniform small-gain condition holds:*

$$\exists \eta \in \mathcal{K}_\infty : \qquad \operatorname{dist}\left(A(x) - x, \mathbb{R}^n_+\right) \geq \eta(\|x\|_X) \quad \forall x \in \mathbb{R}^n_+.$$

3. $\exists \eta \in \mathcal{K}_\infty$: $\quad A(x) \not\geq x - \eta(\|x\|_X)\mathbf{1} \quad \forall x \in \mathbb{R}^n_+ \backslash \{0\}$.

If $A = \Gamma_\otimes$, *then above conditions are equivalent to*

4. *A satisfies the strong small-gain condition:*

$$\exists \rho \in \mathcal{K}_\infty : \qquad (\mathrm{id} + \rho) \circ A(x) \not\geq x, \quad x \in \mathbb{R}^n_+ \backslash \{0\}.$$

- The proof of (4) $\Rightarrow$ (3) exploits Lemma 13 in [Dashkovskiy, Rüffer, Wirth, 2007].

Small-gain conditions: nonlinear n-dim case

Proposition (M., Kawan, Glück, 2020)

Assume that $(X, X^+) = (\mathbb{R}^n, \mathbb{R}^n_+)$ *for some* $n \in \mathbb{N}$, *and* $A : X^+ \to X^+$ *be continuous. TFAE:*

1. *A has monotone limit property.*
2. *The uniform small-gain condition holds:*
$$\exists \eta \in \mathcal{K}_\infty : \qquad \mathrm{dist}\,(A(x) - x, \mathbb{R}^n_+) \geq \eta(\|x\|_X) \quad \forall x \in \mathbb{R}^n_+.$$
3. $\exists \eta \in \mathcal{K}_\infty$: $\quad A(x) \not\geq x - \eta(\|x\|_X)\mathbf{1} \quad \forall x \in \mathbb{R}^n_+ \backslash \{0\}$.

If $A = \Gamma_\otimes$, *then above conditions are equivalent to*

4. *A satisfies the strong small-gain condition:*
$$\exists \rho \in \mathcal{K}_\infty : \qquad (\mathrm{id} + \rho) \circ A(x) \not\geq x, \quad x \in \mathbb{R}^n_+ \backslash \{0\}.$$

- The proof of (4) $\Rightarrow$ (3) exploits Lemma 13 in [Dashkovskiy, Rüffer, Wirth, 2007].

- Strong small-gain condition for $\Gamma_\otimes$ can be efficiently checked via cyclic conditions.

Small-gain condition for couplings of 2 systems

If we have only 2 systems, the gain operator takes form

$$\Gamma_{\otimes}(s) = \begin{pmatrix} \gamma_{12}(s_2) \\ \gamma_{21}(s_1) \end{pmatrix}.$$

and the strong small-gain condition

$$\exists \rho \in \mathcal{K}_\infty : \qquad (\mathrm{id} + \rho) \circ \Gamma_{\otimes}(x) \not\geq x, \quad x \in \mathbb{R}^n_+ \backslash \{0\}.$$

takes form

$$\exists \rho \in \mathcal{K}_\infty : \qquad (\mathrm{id} + \rho) \circ \gamma_{12} \circ (\mathrm{id} + \rho) \circ \gamma_{21}(r) < r \quad \forall r > 0,$$

and corresponds to the small-gain condition in [Jiang, Teel, Praly, 1994].

Example: spatially invariant linear system

Consider an infinite interconnection of scalar subsystems for some $a, b > 0$:

$$\dot{x}_i = ax_{i-1} - x_i + bx_{i+1} + u, \quad i \in \mathbb{Z}.$$

Σ is well-posed with $X = \ell_\infty(\mathbb{Z}), \mathcal{U} := L_\infty(\mathbb{R}_+, \mathbb{R})$.

Proposition

$$\Sigma \textit{ is ISS} \quad \Leftrightarrow \quad a + b < 1.$$

$\Rightarrow$. $y : t \mapsto (e^{(a+b-1)t}\mathbf{1})_{i \in \mathbb{Z}}$ solves the system for initial condition $\mathbf{1}$ and $u \equiv 0$. Thus, if $a + b \geq 1$, Σ is not ISS.

Example: spatially invariant linear system

Consider an infinite interconnection of scalar subsystems for some $a, b > 0$:

$$\dot{x}_i = ax_{i-1} - x_i + bx_{i+1} + u, \quad i \in \mathbb{Z}.$$

Σ is well-posed with $X = \ell_\infty(\mathbb{Z}), \mathcal{U} := L_\infty(\mathbb{R}_+, \mathbb{R})$.

Proposition

$$\Sigma \text{ is ISS} \quad \Leftrightarrow \quad a + b < 1.$$

$\Rightarrow$. $y : t \mapsto (e^{(a+b-1)t}\mathbf{1})_{i\in\mathbb{Z}}$ solves the system for initial condition $\mathbf{1}$ and $u \equiv 0$. Thus, if $a + b \geq 1$, Σ is not ISS.
$\Leftarrow$. Let $a + b < 1$. We have:

$$|x_i(t)| \quad \leq \quad e^{-t}|x_i(0)| + a\|x_{i-1}\|_\infty + b\|x_{i+1}\|_\infty + \|u\|_\infty.$$

Define

$$\Gamma(s) = (as_{i-1} + bs_{i+1})_{i\in\mathbb{Z}}, \quad s = (s_i)_{i\in\mathbb{Z}} \in \ell_\infty^+(\mathbb{Z}).$$

We have:

$$\|\Gamma\| \leq a + b < 1.$$

Hence $r(\Gamma) < 1$, and the network is ISS.

Example: spatially invariant nonlinear system

Consider an infinite interconnection of scalar subsystems for some $a, b > 0$:

$$\dot{x}_i = -x_i^3 + \max\{ax_{i-1}^3, bx_{i+1}^3, u\}, \quad i \in \mathbb{Z}.$$

Σ is well-posed with $X = \ell_\infty(\mathbb{Z}), \mathcal{U} := L_\infty(\mathbb{R}_+, \mathbb{R})$.

Proposition

$$\Sigma \text{ is ISS} \quad \Leftrightarrow \quad a < 1 \ \wedge \ b < 1.$$

Let $a, b < 1$. Then there are $\beta \in \mathcal{KL}$ and $a_1, b_1 < 1$ such that:

$$|x_i(t)| \leq \beta(|x_i(0)|, t) + \max\{a_1\|x_{i-1}\|_\infty, b_1\|x_{i+1}\|_\infty\} + (1+\varepsilon)^{1/3}\|u\|_\infty^{1/3},$$

Define $\Gamma : \ell_\infty^+(\mathbb{Z}) \to \ell_\infty^+(\mathbb{Z})$ as

$$\Gamma(s) = (\max\{a_1 s_{i-1}, b_1 s_{i+1}\})_{i\in\mathbb{Z}}, \quad s = (s_i)_{i\in\mathbb{Z}} \in \ell_\infty^+(\mathbb{Z}).$$

Γ satisfies $\lim_{n\to\infty}\left(\sup_{j_1,\ldots,j_n} \gamma_{j_1 j_2}\cdots\gamma_{j_{n-1}j_n}\right)^{1/n} < 1$, which implies MLIM property.
Small-gain theorem implies ISS of the infinite network.

Take-Home Slide

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let

1. $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, *$i \in \mathbb{N}$ be ISS*
2. $\Sigma = (X, \mathcal{U}, \phi)$ *be well-defined*
3. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$: $\quad \beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad r \in \mathbb{R}_+,\ t \geq 0,\ i \in \mathbb{N}.$
4. $\Gamma_\otimes$ *has monotone limit property*

Then Σ is ISS.

Highlights

- Finite and infinite networks
- Subsystems of any type and dimension (ODEs, PDEs, delay systems, etc.)
- Couplings of any type (in-domain or boundary couplings)
- Mild requirements on regularity.
- New already for finite networks of time-delay systems.
- For finite networks of ODEs it recovers (even under less regularity assumptions on f) Dashkovskiy, Rüffer, Wirth. *An ISS small gain theorem for general networks*, MCSS, 2007.

Take-Home Slide

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let

1. $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ *be ISS*
2. $\Sigma = (X, \mathcal{U}, \phi)$ *be well-defined*
3. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$*:* $\quad \beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad r \in \mathbb{R}_+,\ t \geq 0,\ i \in \mathbb{N}.$
4. $\Gamma_\otimes$ *has monotone limit property*

Then Σ *is ISS.*

Constructive special cases

- Small-gain theorems for Linear gain operators
- Small-gain theorems for Finite networks

Take-Home Slide

Theorem (ISS Small-gain theorem (M., Kawan, Glück, 2020))

Let

1. $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ *be ISS*
2. $\Sigma = (X, \mathcal{U}, \phi)$ *be well-defined*
3. $\exists\, \gamma \in \mathcal{K}$ *and* $\beta \in \mathcal{KL}$: $\quad \beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad r \in \mathbb{R}_+,\ t \geq 0,\ i \in \mathbb{N}.$
4. $\Gamma_\otimes$ *has monotone limit property*

Then Σ *is ISS.*

Constructive special cases

- Small-gain theorems for Linear gain operators
- Small-gain theorems for Finite networks

Further results

- Sum-type ISS small-gain condition for infinite networks
- Small-gain theorems for non-uniform ISS and UGS properties
- Small-gain theorems for compact / sublinear / homogeneous operators

Outlook: Lyapunov small-gain theorems for infinite networks

Linear gain operators

- If the spectral radius of the gain operator $< 1 \quad \Rightarrow \quad$ Network is ISS.
- Applications:
  - Time-varying infinite networks
  - Consensus in infinite-agent systems
  - Design of distributed observers for infinite networks
- References:
  - Kawan, M., Swikir, Noroozi, Zamani. *A Lyapunov-based small-gain theorem for infinite networks.* Submitted to IEEE TAC.
  - Noroozi, M., Kawan, Zamani. *Small-gain theorem for stability, cooperative control and distributed observation of infinite networks.* ArXiv Preprint.

Nonlinear gain operators

- Dashkovskiy, Pavlichkov. *Stability conditions for infinite networks of nonlinear systems and their application for stabilization*. Automatica, 2020.
- Dashkovskiy, M., Schmid, Wirth. *Stability of infinitely many interconnected systems*. NOLCOS, 2019.

Overview

Open problems

- Nonlinear ISS Lyapunov small-gain theorem for infinite networks
- Relation between monotone limit property and uniform small-gain condition.
- Our results are applicable to boundary couplings, but:
  - Well-posedness analysis of PDEs coupled via boundary, is a challenging problem
  - Verifying ISS w.r.t. boundary inputs is a challenging problem, especially for nonlinear systems.

References

- M., Kawan, Glück. *Small-gain theorems for ISS of infinite interconnections*, in preparation, 2020.
- Glück, Kawan, M. *Stability criteria for discrete-time positive linear systems*, in preparation, 2020.
- M. *Small gain theorems for general networks of heterogeneous infinite-dimensional systems.* Submitted to SICON, 2019.
- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* IEEE TAC, 2018.
- M., Ito. *Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach.* SICON, 2015.